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MATERIALS WITH MEMORY

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by

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I

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Materials with Memory

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Abstract

The constitutive equations for materials with memory developed by Green and Rivlin are discussed, with particular emphasis on the physical assumptions regarding material behavior which are implied by the mathematical assumptions in the theory.

1. Introduction

The characteristic property of viscoelastic solids which distinguishes them from perfectly elastic solids is the fact that, if they are subjected to a deformation which varies with time, the stress measured at time t, say, depends not only on the instantaneous value of the deformation gradients, but also on the whole previous history of the deformation gradients. In a series of papers Green and Rivlin (1957, 1960) and Green, Spencer and Rivlin (1959) have developed constitutive equations for such materials, in which the stress is expressed in terms of the deformation in the form of series of multiple integrals. It is the object of this paper to recapitulate this development, with particular emphasis on the physical assumptions regarding the materials which are implied by the mathematical assumptions made in the theory. Such a development is perhaps timely in view of the extensive attempts in recent years to represent the behavior of actual materials in the form given by the theory.

In §2, we start with the assumption that the Cauchy stress is a functional of the deformation gradients for times up to and including t and express the restrictions imposed on the form of this dependence by the consideration that, if the body and the force system applied to it, including inertial forces, are simultaneously subjected to a time-dependent rigid rotation, the stress field at time t is correspondingly rotated by the amount of this rotation at time t . The functional dependence on the history of the deformation gradients up to and including time t is thus replaced, with any desired approximation, by functional dependence on the reduced Cauchy strain (i.e., the Cauchy strain less the unit tensor). The nature of this dependence can be further restricted if the material has some symmetry, but the explicit determination of the form of this restriction will not be discussed here. Rather we shall be concerned with the assumptions regarding the functional dependence of the stress on the reduced Cauchy strain which enable us to obtain explicit representations.

Since much of the apparent complexity of the theory, as originally presented, stems from the tensor character of the equations involved, we first discuss the simple case in which only simple extension of a rod of the material under tensile force is considered. The stress is then considered to be a functional of the history of the fractional extension of the rod up to and including time t . The assumption is made that, for two extension histories, for which the extensions at each instant of time are infinitesimally different, the corresponding stresses at time t are infinitesimally different. This assumption that the dependence of the stress on the extension history is continuous, enables us to obtain, with any desired approximation, representations for the stress at time t as the sum of a series of multiple integrals in the extension history. It is shown that the kernels in these multiple integrals must necessarily be such as to yield the property that the memory of the stress for extension history is a fading memory - the stress forgets the extension history in the distant enough past. This results from the continuity assumption and from the assumption that, whatever the extension history, the stress does not become infinite at infinite time.

It is pointed out that the basic assumption that the stress at time t is a functional of the extension history up to and including time t is strictly not broad enough to accommodate the behavior of materials which have internal friction, if deformations involving instantaneous changes of the velocity gradient are allowed. In such cases the stress in the rod at time t must be assumed to depend not only on the history of the extension up to and including t, but also on the instantaneous values of the rate of extension at time t.

Finally in §4, the various types of behavior considered in §3, in the case of time-dependent simple extension of a rod, are generalized to the case of arbitrary deformations of the material.

2. Basic theory

In this section, the principles underlying the continuum mechanical theory of non-linear viscoelastic solids will be briefly summarized. We start by delimiting in mathematical terms the class of materials to which the theory is to apply. This class is not the most general with which we shall be concerned in this paper. However, it will serve to illustrate the fundamental principles involved.

The deformation is described by specifying the vector position $\underset{\sim}{x}(\tau)$ of a generic particle of the body at time τ, with respect to a fixed origin, as a function of its vector position $\underset{\sim}{X}$ with respect to the same origin at some reference time T, say, which is conveniently taken to be a time at which the body is undeformed. Thus,

$$\underset{\sim}{x}(\tau) = \underset{\sim}{x}(\underset{\sim}{X},\tau) . \tag{2.1}$$

The displacement vector $\underset{\sim}{u}(\tau)$, defined by

$$\underset{\sim}{u}(\tau) = \underset{\sim}{x}(\tau) - \underset{\sim}{X} , \tag{2.2}$$

may also be regarded as a function of $\underset{\sim}{X}$ and τ, thus

$$\underset{\sim}{u}(\tau) = \underset{\sim}{u}(\underset{\sim}{X},\tau) . \tag{2.3}$$

In terms of the components $x_p(\tau)$, X_A and $u_p(\tau)$ of $\underset{\sim}{x}(\tau)$, $\underset{\sim}{X}$ and $\underset{\sim}{u}(\tau)$ respectively, in a rectangular cartesian coordinate system x, the relation (2.1) may be written

$$x_p(\tau) = x_p(X_A,\tau) \tag{2.4}$$

and the relation (2.3) may be written

$$u_p(\tau) = u_p(X_A, \tau) . \tag{2.5}$$

It is evident that if the nine deformation gradients $\partial x_p(\tau)/\partial X_A$ are specified at a point, then the amount by which any linear element at that point is stretched is determined. For brevity, we use the so-called comma notation, in which $_{,A}$ denotes the operator $\partial/\partial X_A$. The nine deformation gradients $\partial x_p(\tau)/\partial X_A$ are then written $x_{p,A}(\tau)$.

We shall be concerned with materials in which the stress $\sigma_{ij}(t)$ at a generic particle, measured at time t, which, for brevity, will be denoted σ_{ij} , may depend on the history of the deformation gradients $x_{p,A}(\tau)$ at that particle for all times from $\tau = -$ to t . Thus, unlike the situation which exists in an elastic material, in which the stress at time t is determined uniquely by the deformation gradients at time t , in a viscoelastic material the stress at time t "remembers" the deformation gradients at all previous times and the material is said to possess "memory". We can express this physical idea in a mathematical statement - the <u>constitutive assumption</u> for the material - that the stress σ_{ij} at time t is a tensor-valued <u>functional</u> of the deformation gradient history $x_{p,A}(\tau)$ for $\tau = -\infty$ to t, thus

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=-\infty}^{t}[x_{p,A}(\tau)] . \tag{2.6}$$

This means simply that if the dependence of $x_{p,A}(\tau)$ on τ is known for a generic particle, then the stress at that particle, at time t, is determined.

We recall that, in classical elasticity theory, if we make the constitutive assumption that the stress at time t depends on the values of the nine displacement gradients $\partial u_i/\partial x_j$ at time t, it is shown that it must in fact do so through the six infinitesimal strain components e_{ij} defined by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \tag{2.7}$$

The question arises - are there any corresponding restrictions which, in the case of the constitutive assumption (2.6) for a non-linear viscoelastic material, can be placed on the manner in which σ_{ij} depends on the deformation gradient history? That such restrictions do, in fact, exist follows from a consideration somewhat similar to that used in classical elasticity theory. We consider that the body, together with the force system associated with it,including inertial forces, is subjected to a time-dependent rigid rotation. The stress field at time t, referred to the system x, differs from that which obtains in the absence of the superposed rotation,only to the extent that it undergoes a rotation, the amount of which is that of the superposed rotation at time t. From this consideration, it follows, by a purely mathematical argument, that the dependence of σ_{ij} on the deformation gradient history must be of the following form

$$\sigma_{ij} = x_{i,P}\, x_{j,Q} \mathop{\mathcal{F}_{PQ}}_{\tau=-\infty}^{t} [E_{AB}(\tau)] \ , \tag{2.8}$$

where $E_{AB}(\tau)$ is defined by

$$E_{AB}(\tau) = x_{k,A}(\tau)x_{k,B}(\tau) - \delta_{AB} \ , \tag{2.9}$$

and the abbreviation $x_{i,P}$ is used for the deformation gradients $x_{i,P}(t)$ at time t. The nine quantities $E_{AB}(\tau)$ are called the components of the reduced Cauchy strain tensor at time τ and we note that $E_{AB}(\tau) = E_{BA}(\tau)$. Accordingly, only six of the quantities $E_{AB}(\tau)$ are independent. Thus, in passing from (2.6) to (2.8), we have replaced arbitrary dependence of σ_{ij} on the nine functions $x_{p,A}(\tau)$ by arbitrary dependence on the six independent components of the reduced Cauchy strain.

So far we have made no assumption regarding the symmetry of the material. We recall that in classical elasticity, if the material possesses some symmetry, restrictions can be placed on the manner in which the stress depends on the infinitesimal strain components. In the case when the material is isotropic these restrictions become particularly strong.

In principle, we can, by similar considerations, place restrictions on the form of the tensor-valued functional $\mathcal{F}_{PQ}$ in (2.8) if the non-linear viscoelastic material has some symmetry. It emerges that the functional $\mathcal{F}_{PQ}$ must satisfy the condition

$$\mathop{\mathcal{F}_{PQ}}_{\tau=-\infty}^{t}[\bar{E}_{AB}(\tau)] = a_{PM}a_{QN} \mathop{\mathcal{F}_{MN}}_{\tau=-\infty}^{t}[E_{AB}(\tau)] \ , \tag{2.10}$$

where

$$\bar{E}_{AB}(\tau) = a_{AM}a_{BN}E_{MN}(\tau) , \tag{2.11}$$

for all a_{AM} belonging to the group of transformations describing the symmetry of the material. In the case when the material is isotropic, this group is the rotation group, so that, for an isotropic viscoelastic material, the tensor-valued functional F_{AB} must satisfy the condition (2.10) for all rotations a_{AM}. This implicit restriction on the form of F_{AB} can be made explicit for any specified material symmetry by a method developed by Green and Rivlin (1957) and Wineman and Pipkin (1964). However, we shall not pursue this here.

If the material considered is incompressible, specification of the deformation gradient history does not determine the stress completely, since the superposition on any force system of a hydrostatic pressure does not alter the deformation. Accordingly, the constitutive assumption (2.6) must be replaced by

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=-\infty}^{t}[x_{p,A}(\tau)] - p\delta_{ij} , \tag{2.12}$$

where p is undetermined if the deformation gradient history is specified. From this constitutive assumption the constitutive equation

$$\sigma_{ij} = x_{i,P}x_{j,Q} \mathop{F_{PQ}}_{\tau=-\infty}^{t}[E_{AB}(\tau)] - p\delta_{ij} \tag{2.13}$$

follows, replacing (2.8). At the same time, the fact that, in an incompressible material, the volume of a material element must remain unchanged during deformation, imposes on the reduced Cauchy strain the restriction

$$\det |E_{AB}(\tau) + \delta_{AB}| = 1 . \tag{2.14}$$

There is one further restriction which can be placed on the constitutive equation (2.8) or (2.13) from considerations of a very general character. This involves the concept of a <u>hereditary</u> material as one which remains unchanged in properties so long as it rests undeformed. For such a material a shift in time, by an amount t_0, say, of the deformation history, will shift the dependence of stress vs. time curve by an equal amount t_0 parallel to the time axis.

In the remainder of this paper, we shall restrict ourselves to hereditary materials.

3. Non-linear constitutive equations

In §2 we have obtained certain restrictions which must be imposed on the manner in which the Cauchy stress depends on the deformation gradient history. These are expressed by equation (2.8) in which, if the material possesses some symmetry, the tensor-valued functional F_{PQ} must satisfy the equation (2.10).

In order to obtain more explicit forms for the constitutive equation, which reflect physically non-pathological behavior on the part of the material described, various representations of the functional F_{PQ} in terms of series of multiple integrals have been developed. Green and Rivlin (1957, 1960) and Green, Rivlin and Spencer (1959) considered materials in which infinitesimal changes in the deformation history result in infinitesimal changes in the stress at time t. They showed that a sequence of approximations to F_{PQ} in the form of series of multiple integrals can be constructed which tend, in the limit, to F_{PQ}. In order to highlight the physical content of this development, free from the mathematical complications which arise from the tensorial character of the constitutive equations, we will discuss it in the context of time-dependent simple extension of a rod.

Accordingly, let σ be the stress at time t which results from a time-dependent fractional extension $e(\tau)$. Then, since σ is determined if $e(\tau)$ is specified for $-\infty<\tau\leq t$, we may say that σ is a functional of $e(\tau)$, thus:

$$\sigma = \mathop{F}_{\tau=-\infty}^{t} [e(\tau)] \ . \tag{3.1}$$

It is convenient for the purposes of our discussion to replace the time τ by a time-like variable s defined by

$$s = \frac{1}{(t-\tau+1)^{\rho}} \ , \tag{3.2}$$

where ρ is positive. Then, for a specified t, the extension may be regarded as a function of s rather than of τ. Noting that when $\tau = -\infty$, $s = 0$ and when $\tau = t$, $s = 1$, we may re-write (3.1), for a specified t, in the form

$$\sigma = \mathop{F}_{s=0}^{1} [e(s);t] \ , \tag{3.3}$$

σ being a functional of $e(s)$ and an ordinary function of t. For hereditary materials, we may omit the dependence of σ on t and we then have

$$\sigma = \mathop{F}_{s=0}^{1} [e(s)] \ . \tag{3.4}$$

Now, suppose that $e(s)$ can be represented by a Fourier series (see §5) thus

$$e(s) = \sum_{n=0}^{\infty} A_n \cos n \Pi s \ , \tag{3.5}$$

where

$$A_n = 2\int_0^1 e(s)\cos n\Pi s\, ds\ ,$$
$$A_0 = \int_0^1 e(s)ds\ , \tag{3.6}$$

The condition, under which the representation (3.5) for $e(s)$ is valid, is that $e(s)$ have bounded variation in the range $s = 0, 1$. If this condition is satisfied, then the sum of the series (3.5) gives the value of $e(s)$ at all points of the interval $s = 0, 1$ except possibly at points of discontinuity. Accordingly, if we assume that $e(s)$ is a continuous function of s, then, if the coefficients $A_0, A_1, \ldots$ in the Fourier series (3.5) are specified, the value of $e(s)$ is determined at all points of the interval $s = 0, 1$ and σ may be regarded as a function of $A_0, A_1, \ldots,$ rather than as a function of $e(s)$.

We now suppose that in (3.4), σ is a <u>continuous</u> functional of $e(s)$; i.e., an infinitesimal change*

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* More precisely, if $e_1(s)$ and $e_2(s)$ are two extension histories, and σ_1 and σ_2 are the corresponding values of stress at time t, then for $\varepsilon > 0$, there exists a value of $\delta > 0$, such that

$$|\sigma_1 - \sigma_2| < \varepsilon$$

provided that

$$\max |e_1(s) - e_2(s)| < \delta\ .$$

in $e(s)$ results in an infinitesimal change in σ . Then, σ may be regarded as a continuous function of $A_0, A_1, \ldots$. To some degree of approximation, we may regard σ as a continuous function of the first $N + 1$ Fourier coefficients $A_0, A_1, \ldots, A_N$. This approximation may be made as close as we please by taking N sufficiently large. We denote by σ_N such an approximation to σ . Now, from Weierstrass's theorem, it follows that we can approximate σ_N as closely as we please by a polynomial in $A_0, A_1, \ldots, A_N$. Accordingly, we can approximate σ with any desired accuracy by a polynomial in $A_0, A_1, \ldots, A_N$. We note from (3.6) that the product of r , say, A's is an r-tuple integral and that the kernel in this integral is a continuous function of its arguments. For example ,

$$A_1 A_2 = 4\int_0^1\int_0^1 \cos \Pi s_1 \cos 2\Pi s_2 \; e(s_1)e(s_2)ds_1 ds_2 \; . \qquad (3.7)$$

Accordingly, σ may be approximated to any desired accuracy by the sum of a number of multiple integrals, thus

$$\sigma = \sum_{\mu} \int_0^1 \cdots \int_0^1 f_\mu(s_1, s_2, \ldots, s_\mu) e(s_1) e(s_2) \ldots e(s_\mu) \, ds_1 ds_2 \ldots ds_\mu = \bar{\sigma}(\text{say}), \qquad (3.8)$$

where $\mu = 0, 1, 2, \ldots$, and the kernels in the multiple integrals are continuous functions of $s_1, s_2, \ldots, s_\mu$.

The term in (3.8), which corresponds to $\mu = 0$, is a constant. This may be taken to be zero, if we assume that the stress is zero for zero strain history (i.e., $\sigma = 0$ when $e(s)=0$). The result (3.8) is that obtained from the theory of Green and Rivlin (1957), by specializing it to the case of simple extension of a rod.

Using (3.2), we can rewrite the expressions (3.6) for A_n as

$$A_n = 2\int_{-\infty}^{t} \frac{\rho}{(t-\tau+1)^{\rho+1}} e(\tau) \cos \frac{n\Pi}{(t-\tau+1)^{\rho}} d\tau ,$$
$$A_0 = \int_{-\infty}^{t} \frac{\rho}{(t-\tau+1)^{\rho+1}} e(\tau) d\tau . \tag{3.9}$$

Correspondingly, the approximate expression for σ in (3.8) may be rewritten as

$$\sigma \approx \bar{\sigma} = \sum_{\mu} \int_{-\infty}^{t} \bar{f}_{\mu}(\tau_1, \tau_2, \ldots, \tau_{\mu}) \tag{3.10}$$
$$e(\tau_1)e(\tau_2)\ldots e(\tau_{\mu})d\tau_1 d\tau_2 \ldots d\tau_{\mu} ,$$

where

$$\bar{f}_{\mu}(\tau_1, \tau_2, \ldots, \tau_{\mu}) = \frac{\rho^{\mu} f_{\mu}(s_1, s_2, \ldots, s_{\mu})}{[(t-\tau_1+1)(t-\tau_2+1)\ldots(t-\tau_{\mu}+1)]^{\rho+1}} . \tag{3.11}$$

Since $f_{\mu}(s_1, s_2, \ldots, s_{\mu})$ is a bounded function of $s_1, s_2 \ldots, s_{\mu}$, and hence of $\tau_1, \tau_2, \ldots, \tau_{\mu}$, it follows that $\bar{f}_\mu(\tau_1,\tau_2,\ldots,\tau_\mu)$ becomes vanishingly small as any of the variables $\tau_1, \tau_2,\ldots,\tau_\mu$ tends to $-\infty$. This implies that the material has fading memory in the usual sense that, as $t - \tau \to \infty$, the effect on σ of the extension, in a finite time interval about time τ, becomes vanishingly small.

From a purely mathematical point of view, if $e(s)$ possesses one or more discontinuities in the interval $s = 0,1$, then in order to determine $e(s)$ at all points of this interval, we must specify, not only the values of the Fourier coefficients given by (3.6), but also the values of $e(s)$ at the various values of s at which the discontinuities occur. Let us suppose that discontinuities occur at $s = S_\lambda (\lambda=1,2,\ldots,\nu)$ and let $e_\lambda = e(S_\lambda)$. Then, if σ is a continuous functional of $e(s)$, we may regard it as a continuous function of $A_0, A_1,\ldots$ and of $e_\lambda (\lambda=1,2,\ldots,\nu)$, the nature of this function depending on the values S_λ of s at which the discontinuities occur. Approximate expressions $\bar{\sigma}$ for σ may then be obtained in the form (3.8), where f_μ is also a function of e_λ and S_λ, as well as of $s_1,s_2,\ldots,s_\mu$. The term corresponding to $\mu = 0$ is now a function of e_λ and S_λ.

In considering the applicability of this conclusion to real materials, it is well to bear in mind that discontinuous changes in the extension cannot in fact be produced. The importance of considering a discontinuous change in $e(s)$ at time $s = S_1$, say, resides only in the fact that it provides an idealized model for a very rapid, but continuous, change in the value of $e(s)$ in the interval $S_1 - \varepsilon$, $S_1 + \varepsilon$, where ε is small. Consequently, we may, in general, omit consideration of discontinuities in $e(s)$. An exception arises when this discontinuity occurs at the instant at which the stress is measured, i.e., at $s = 1$. In order to accommodate perfectly elastic materials and materials exhibiting instantaneous elasticity within the framework of the theory, we must then include, in the expression for σ , explicit dependence on the instantaneous value e of $e(s)$ at $s = 1$. Approximate expressions $\bar{\sigma}$ for σ may then be obtained in the form (3.8), where f_μ is a function of e , as well as of $s_1, s_2, \ldots, s_\mu$. The term corresponding to $\mu = 0$ is now a function of e . This result is essentially that obtained from the theory of Green, Rivlin and Spencer (1959), by specializing it to the case of simple extension of a rod.

It is perhaps worth-while to underline the fact that, whether or not the f's depend on e , the approximation to σ represented by (3.8) is of the following type. If ε is any specified positive quantity, we can construct an approximation $\bar{\sigma}$, say, to σ of the form (3.8), such that

$$|\sigma - \bar{\sigma}| < \varepsilon \tag{3.12}$$

for all extension histories $e(s)$ which have bounded variation.

The kernels occuring in the expression for $\bar{\sigma}$ cannot be uniquely determined. We can, however, construct a sequence of approximations of the form (3.8), corresponding to smaller and smaller ε , which tends in the limit to σ . Corresponding kernels in this sequence do not necessarily tend to a limit so that we <u>cannot</u> say that σ may, <u>without error</u>, be represented by an expression of a form similar to that in (3.8). The <u>absolute</u> error involved in approximating σ by $\bar{\sigma}$ may, however, be made as small as we please. In the neighborhood of $\sigma = 0$, the <u>percentage</u> error may then be large.

It is, however, perfectly safe to use a representation for σ of the form (3.8),provided that we are concerned only with changes of σ of magnitude much greater than ε . Since ε may be made as small as we please, it might, at first sight, appear that this does not impose a meaningful restriction on the use of the representation (3.8). Let us therefore consider an example in which this might, in fact, impose a meaningful restriction. Suppose we consider extension histories $\alpha e(s)$, with e zero, where α may be made as small as we please. For small enough α , and taking $f_0 = 0$, we may approximate $\bar{\sigma}$ by the first term in the series (3.8) thus

$$\bar{\sigma} = \alpha \int_0^1 f_1(s_1) e(s_1) ds_1 \ . \qquad (3.13)$$

This approximation becomes increasingly good as as α decreases. However, regarded as an approximation to σ , the expression on the right-hand side of (3.13) has the limitation that σ only approximates $\bar{\sigma}$ with absolute error less than ε . It may be that, in order for (3.13) to provide a good approximation to σ , the value of α must be so decreased that the stress is comparable with ε . The fact that ε may be made as small as we please does not save us from this limitation, since choice of a lower value of ε requires a new approximate representation of the form (3.8) and with this we may be driven to lower values of α and hence of the stress.

If, however, σ may be expressed exactly in the form (3.8), then, for sufficiently small values of α , we may approximate σ by the expression (3.13) and, for somewhat larger values, by

$$\sigma = \alpha\int_0^1 f_1(s_1)e(s_1)ds_1 + \alpha^2\int_0^1\int_0^1 f_2(s_1,s_2) e(s_1)e(s_2)ds_1ds_2 , \qquad (3.14)$$

and so on. In this way, we obtain a hierarchy of expressions for σ , valid over larger and larger ranges of the extension prior to time t . This is, of course, also true if σ can be approximated by an expression of the form (3.8), with an error which is of degree greater than μ , say, in the extension. Of course, whether or not one of the approximations has significant value in the study of a particular material, to which it applies in principle, will depend on whether the range of extensions for which it is valid is a range over which experiments can be carried out with reasonable accuracy.

Coleman and Noll (1960) have obtained n successive approximations essentially of the forms (3.13) and (3.14) by assuming that the functional F in (3.3) is Fréchet differentiable n times about the zero history (i.e., the history $e(s) = 0$, $0 \le s \le 1$). Their argument is basically circular in that the definition of nth order Fréchet differentiability is that the functional F shall be expressible as the sum of homogeneous functionals of degrees $1, 2, \ldots, n$ in $e(s)$, with a residue of degree greater than n in $e(s)$.

The discussion has, so far, been based on the assumption that the stress at time t is a continuous functional of the extension history $e(\tau)$ for $-\infty < \tau \le t$. It is evident that this is not necessarily an appropriate assumption for many of the materials with which we are concerned. For example, let us consider two deformations in both of which the extension is maintained at zero up to and including time t . In one of these, we maintain the extension zero after time t , while in the other the rate of extension is changed discontinuously at time t from zero to a finite value. In many materials (e.g., in Newtonian fluids) the stress at time t will be quite different in the two cases.

In order to accommodate such materials in the framework of our mathematics, we may take our constitutive assumption in the following form : the stress is a functional of $e(\tau)$ up to and including time t and an ordinary function of $\dot{e}$ (the rate of extension at time t) , thus :

$$\sigma = F\,[e(\tau)\ ;\ \dot{e}]\ . \tag{3.15}$$

Generalizing this concept to accommodate materials for which discontinuous changes at time t in the higher time derivatives $\ddot{e}$, $\dddot{e}$,... of the extension affect the stress at time t , we may make our initial constitutive assumption in the form : the stress at time t is a functional of $e(\tau)$ in the interval $-\infty < \tau \leq t$ and an ordinary function of $\dot{e}$, $\ddot{e}$,..., thus

$$\sigma = \mathop{F}_{\tau=-\infty}^{t} [e(\tau) ; \dot{e}, \ddot{e}, \ldots] . \qquad (3.16)$$

This is esentially the constitutive assumption used by Green and Rivlin (1960), specialized to the case of simple extension of a rod.

Alternatively, we could accommodate such constitutive assumptions in the single assumption that the stress σ at time t is given by

$$\sigma = \mathop{L^t}_{\varepsilon\to 0} \mathop{F}_{\tau=-\infty}^{t+\varepsilon} [e(\tau)] , \qquad (3.17)$$

for small positive ε , i.e., stress at time t is the limit as $\varepsilon \to 0$ of a functional of $e(\tau)$ over the range $-\infty < \tau \leq t + \varepsilon$. To see this we note that

$$\begin{aligned} \dot{e} &= - \mathop{L^t}_{\varepsilon\to 0} \int_{-\infty}^{t+\varepsilon} \delta'(\tau-t)e(\tau)d\tau , \\ \ddot{e} &= \mathop{L^t}_{\varepsilon\to 0} \int_{-\infty}^{t+\varepsilon} \delta''(\tau-t)e(\tau)d\tau , \quad \text{etc.,} \end{aligned} \qquad (3.18)$$

where $\delta(\ )$ denotes the Dirac delta function and $\delta'(\ )$ denotes its derivative. It should be noted that even if, in (3.15) and (3.16), σ is a continuous functional of $e(\tau)$ and a continuous function of the instantaneous values of the time derivatives of e at time t, the functional dependence on $e(\tau)$ expressed by (3.17) is not, in general, continuous.

Of course, if we limit ourselves to deformations in which the time derivatives of $e(\tau)$ are continuous at t, then the constitutive assumptions (3.15) and (3.16) can be replaced by (3.1).

Approximations to σ in the form of series of multiple integrals may be made in the cases when σ is given by (3.15) or (3.16). The approximation takes the form (3.10), with the kernels dependent on $\dot{e}$ in the case (3.15) and on $\dot{e}$, $\ddot{e}$,... in the case (3.16).

It should be appreciated, in interpreting the above remarks, that if the "slope" of $e(\tau)$ vs. τ changes discontinuously at time t, then strictly its time derivative at time t does not exist. By $\dot{e}$ we mean the right-hand derivative of $e(\tau)$ at time t. This is in accord with the usual convention in mechanics when we apply the Navier-Stokes equation, say, to situations in which the rate of deformation changes discontinuously at time t.

4. Generalization to arbitrary deformations

In the previous section, we have attempted to show how mathematical expression may be given to a variety of possible types of mechanical behavior which may be exhibited by viscoelastic materials. We have done this in the relatively simple context of simple extension.

For each type of behavior we may obtain analogous equations which are valid for deformations of a general character. Thus, if we take our initial constitutive assumption in the form (2.6), we have seen that the Cauchy stress must necessarily be expressible in the form (2.8). Replacing the time τ by the time-like variable s, defined by (3.2), we see that the stress σ_{ij}, referred to a rectangular Cartesian system x, must be expressible in the form

$$\sigma_{ij} = x_{i,P}x_{j,Q} \mathop{\mathcal{F}_{PQ}}_{s=0}^{1} [E_{PQ}(s)] , \tag{4.1}$$

where $E_{PQ}(s)$ is the reduced Cauchy stress at time s. This can be re-written more succinctly in matrix notation, thus :

$$\underset{\sim}{\sigma} = \underset{\sim}{F} \mathop{\mathcal{F}}_{s=0}^{1} [\underset{\sim}{E}(s)]\underset{\sim}{F}^{T} , \tag{4.2}$$

where

$$\underset{\sim}{\sigma} = ||\sigma_{ij}|| , \quad \underset{\sim}{F} = ||F_{iA}|| = ||x_{i,A}|| , \quad \underset{\sim}{E}(s) = ||E_{AB}(s)|| \tag{4.3}$$

and $\underset{\sim}{F}^T$ denotes the transpose of $\underset{\sim}{F}$.

We now suppose that $\underset{\sim}{E}(s)$ can be represented by a Fourier series thus

$$\underset{\sim}{E}(s) = \sum_{n=0}^{\infty} \underset{\sim}{A}_n \cos n\Pi s , \tag{4.4}$$

where $\underset{\sim}{A}_n$ is given by

$$\begin{aligned} \underset{\sim}{A}_n &= 2\int_0^1 \underset{\sim}{E}(s) \cos n\Pi s \, ds \quad (n \geq 1) \\ \underset{\sim}{A}_0 &= \int_0^1 \underset{\sim}{E}(s) ds ; \end{aligned} \tag{4.5}$$

i.e., each of the components of the tensor $\underset{\sim}{E}(s)$ can be expressed as a Fourier series. This is possible under the same conditions as apply in our discussion of the case of simple extension. Each of the components of $\underset{\sim}{E}(s)$ must have bounded variation in the interval $[0,1]$. Paralleling the discussion in §3, we assume that each of the components of $\underset{\sim}{E}(s)$ is continuous everywhere, except possibly at $s = 1$. Then, in order to represent $\underset{\sim}{E}(s)$ over the complete range $s = [0,1]$, we must specify not only the coefficients $\underset{\sim}{A}_n$ in (4.4), but also the value $\underset{\sim}{E}$ of $\underset{\sim}{E}(s)$ at $s = 1$.

The tensor $\underset{\sim}{F}$ in (4.2) is, of course, a tensor-valued functional and we suppose that it is such that infinitesimal changes in the components of $\underset{\sim}{E}(s)$ lead only to infinitesimal changes in the components of $\underset{\sim}{F}$. The tensor-valued functional $\underset{\sim}{F}$ is then said to be a continuous functional of the tensor $\underset{\sim}{E}(s)$. Analogously with the case of simple extension, we may approximate each of the components of $\underset{\sim}{F}$ to any desired accuracy by a polynomial in the components of the tensors $\underset{\sim}{E}$ and $\underset{\sim}{A}_n$. Thus, each of the components F_{PQ} of $\underset{\sim}{F}$ may be expressed, with any desired accuracy, in the form

$$\begin{aligned} F_{PQ} &\approx \sum_{\mu} \alpha_{PQA_1B_1\ldots A_\mu B_\mu} A^{(n_1)}_{A_1B_1} A^{(n_2)}_{A_2B_2} \ldots A^{(n_\mu)}_{A_\mu B_\mu} , \\ &= \bar{F}_{PQ} \quad \text{(say)} , \end{aligned} \tag{4.6}$$

where, for specified n, $A^{(n)}_{AB}$ are the components of $\underset{\sim}{A}_n$; also $n_1, n_2, \ldots$ take integral values and the α's are functions (or, if we like, polynomial functions) of the components of $\underset{\sim}{E}$.

Using (4.5), we may rewrite (4.6) in the form

$$\begin{aligned} F_{PQ} \approx \bar{F}_{PQ} = \sum_{\mu} \int_0^1 \cdots \int_0^1 & f_{PQA_1B_1\ldots A_\mu B_\mu}(s_1, \ldots s_\mu) \\ & E_{A_1B_1}(s_1) \ldots E_{A_\mu B_\mu}(s_\mu) ds_1 \ldots ds_\mu , \end{aligned} \tag{4.7}$$

where the kernels $f_{PQA_1B_1\ldots A_\mu B_\mu}$ are continuous functions of $s_1, \ldots, s_\mu$ and functions (or polynomial functions) of the components E_{AB} of $\underset{\sim}{E}$.

Using (3.2), we can, of course, re-write (4.7) as

$$F_{PQ} \approx \bar{F}_{PQ} = \sum_{\mu} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \bar{f}_{PQA_1B_1 \cdot \cdot A_\mu B_\mu}(\tau_1, \ldots, \tau_\mu)$$

$$E_{A_1B_1}(\tau_1) \ldots E_{A_\mu B_\mu}(\tau_\mu) d\tau_1 \ldots d\tau_\mu \ , \qquad (4.8)$$

where

$$\bar{f}_{PQA_1B_1 \ldots A_\mu B_\mu} = \frac{\rho^\mu f_{PQA_1B_1 \ldots A_\mu B_\mu}}{[(t-\tau_1+1) \ \ldots (t-\tau_\mu+1)]^{\rho+1}} \qquad (4.9)$$

If the material considered has some symmetry, the functional $\underset{\sim}{F}$ must satisfy the condition (2.10). In approximating $\underset{\sim}{F}$ by $\underset{\sim}{\bar{F}}$, to any desired accuracy, this can be done by an expression of the form (4.7) which satisfies the restriction (2.10) imposed by material symmetry.

We can develop constitutive equations for arbitrary deformations corresponding to the simple extensional case presented in (3.16). We start with the constitutive assumption that the Cauchy stress components at time t are functionals of the histories of the deformation gradients $x_{p,A}(\tau)$ for $-\infty < \tau \leq t$ and functions of the velocity gradients $\dot{x}_{p,A}$, acceleration gradients $\ddot{x}_{p,A}$, and so on, at time t . Thus,

$$\sigma_{ij} = F_{ij}[x_{p,A}(\tau) \ ; \ \dot{x}_{p,A} \ , \ \ddot{x}_{p,A} \ , \ldots] \ . \qquad (4.10)$$

Paralleling the passage from (2.6) to (2.8), we reach the conclusion, in this case, that the stress must be expressible in the form

$$\sigma_{ij} = x_{i,P}x_{j,Q}\mathcal{F}_{PQ}[E_{AB}(\tau) \; ; \; \frac{dE_{AB}}{dt} \; , \; \frac{d^2E_{AB}}{dt^2} \; ,\ldots] \; . \quad (4.11)$$

Using the notation (4.3), we can, of course, rewrite (4.10) as

$$\underset{\sim}{\sigma} = \underset{\sim}{F}\underset{\sim}{\mathcal{F}}[\underset{\sim}{E}(s) \; ; \; \frac{d\underset{\sim}{E}}{dt} \; , \; \frac{d^2\underset{\sim}{E}}{dt^2} \; ,\ldots]\underset{\sim}{F}^T \; . \quad (4.12)$$

If the dependence of $\mathcal{F}_{PQ}$ on $E_{AB}(\tau)$ is continuous, then we can approximate $\mathcal{F}_{PQ}$ with any desired accuracy by a series of multiple integrals in the form (4.8), the kernels now being functions of the components of the reduced Cauchy strain and its time derivatives at time t. If the dependence of $\mathcal{F}_{PQ}$ on $\underset{\sim}{E}$, dE_{AB}/dt, $d^2E_{AB}/dt^2,\ldots$ is continuous, then the dependence of the kernels on these may be taken as polynomial, with any desired accuracy.

Just as in the previous cases discussed, if the material considered has some symmetry, the tensor-valued functional in (4.10) must satisfy the restrictions implied by (2.10).

5. Appendix

In order to arrive at the Fourier representation (3.5) for $e(s)$, we proceed in the following manner. We first form a function $\bar{e}(s)$ given by

$$\begin{aligned} \bar{e}(s) &= e(s) \qquad s = [0,1] , \\ \bar{e}(s) &= e(-s) \qquad s = [-1,0] . \end{aligned} \tag{5.1}$$

$\bar{e}(s)$ is thus an even function of s, defined in the range $[-1,1]$ by (5.1).

We now define $\bar{e}(s)$, not just for $s = [-1,1]$, but for all values of s, as a periodic function with periodicity 2, which is given by (5.1) in the range $[-1.1]$. We express this as a cosine Fourier series thus,

$$\bar{e}(s) = \sum_{n=0}^{\infty} A_n \cos n\Pi s , \tag{5.2}$$

where, using the first of equations (5.1),

$$\begin{aligned} A_n &= \int_{-1}^{1} \bar{e}(s) \cos n\Pi s \, ds = 2\int_{0}^{1} e(s) \cos n\Pi s \, ds \quad (n \geq 1) , \\ A_0 &= \frac{1}{2}\int_{-1}^{1} \bar{e}(s) ds = \int_{0}^{1} e(s) ds . \end{aligned} \tag{5.3}$$

In view of the first of equations (5.1), we obtain (3.5).

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